

Gauge Invariance from a Graphical Self-Consistency Criterion

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Abstract. We propose that quantum physics is the continuous approximation of a more fundamental, discrete graph theory (theory X). Accordingly, the Euclidean transition amplitude Z provides a partition function for geometries over the graph, which is characterized topologically by the difference matrix and source vector of the discrete graphical action. The difference matrix and source vector of theory X are related via a graphical self-consistency criterion (SCC) based on the boundary of a boundary principle on a graph ($\partial_1 \cdot \partial_2 = 0$). In this approach, the SCC ensures the source vector is divergence-free and resides in the row space of the difference matrix. Accordingly, the difference matrix will necessarily have a nontrivial eigenvector with eigenvalue zero, so the graphical SCC is the origin of gauge invariance. Factors of infinity associated with gauge groups of infinite volume are excluded in our approach, since Z is restricted to the row space of the difference matrix and source vector. Using this formalism, we obtain the two-source Euclidean transition amplitude over a $(1+1)$ -dimensional graph with N vertices fundamental to the scalar Gaussian theory.

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1. Introduction

Those who emphasize the incompleteness of quantum field theory (QFT) over its successes often focus on the many ad hoc and, for some, troubling “fixes” involved in the practice of QFT ‡. For example, since QFT is independent of overall factors in the transition amplitude, such factors are simply “thrown away” even when these factors

‡ We are focusing on the “textbook variant of QFT.” Fraser, D.: Quantum Field Theory: Underdetermination, Inconsistency, and Idealization. *Philosophy of Science* 74, 536-565 (October 2009).

are infinity as is the case when the volume of the gauge symmetry group in Faddeev-Popov gauge fixing is infinite[1]. In a petition to philosophers of science, Glashow stated[2], “in a sense it really is a time for people like you, philosophers, to contemplate not where we’re going, because we don’t really know and you hear all kinds of strange views, but where we are. And maybe the time has come for you to tell us where we are.” Rovelli went further stating[3], “As a physicist involved in this effort, I wish that the philosophers who are interested in the scientific description of the world would not confine themselves to commenting and polishing the present fragmentary physical theories, but would take the risk of trying to look ahead.”

Of course, ignoring factors of infinity in the transition amplitude Z per Faddeev-Popov gauge fixing is easily understood in terms of (infinitely) over counting gauge degrees of freedom in the classical field being quantized[4], so there is no problem in that respect. We believe the real issue is the fact that QFT involves the quantization of a classical field[5] when one would rather expect QFT to originate independently of classical field theory, the former typically understood as fundamental to the latter. Herein we accept Glashow and Rovelli’s challenges and respond, not philosophically, but mathematically, and propose a new, fundamental origin for QFT. Specifically, we follow the possibility articulated by Wallace[5] that (p 45), “QFTs as a whole are to be regarded only as approximate descriptions of some as-yet-unknown deeper theory,” which he calls “theory X,” and we propose a new discrete path integral formalism over graphs for “theory X” underlying QFT. Accordingly, sources \mathbf{J} , space and time are self-consistently co-constructed per a graphical self-consistency criterion (SCC) based on the boundary of a boundary principle[6] on the graph ($\partial_1 \cdot \partial_2 = 0$). [In a graphical representation of QFT, part of \mathbf{J} represents field disturbances emanating from a source location (Source) and the other part represents field disturbances incident on a source location (sink).] We call this amalgam “spacetimematter.” The SCC constrains the difference matrix and source vector in Z , which then provides the probability for finding a particular source-to-source relationship in a quantum experiment, i.e., experiments which probe individual source-to-source relations (modeled by individual graphical links) as evidenced by discrete outcomes, such as detector clicks. Since, in QFT, all elements of an experiment, e.g., beam splitters, mirrors, and detectors, are represented by interacting sources, we confine ourselves to the discussion of such controlled circumstances where the empirical results evidence individual graphical links. [Hereafter, all reference to “experiments” will be to “quantum experiments.”] In this approach, the SCC ensures the source vector is divergence-free and resides in the row space of the difference matrix, so the difference matrix will necessarily have a nontrivial eigenvector with eigenvalue zero, a formal characterization of gauge invariance. Thus, our proposed approach to theory X provides an underlying origin for QFT, accounts naturally for gauge invariance, i.e., via a graphical self-consistency criterion, and excludes factors of infinity associated with gauge groups of infinite volume, since the transition amplitude Z is restricted to the row space of the difference matrix and source vector.

While the formalism we propose for theory X is only suggestive, the computations are daunting, as will be evident when we present the rather involved graphical analysis underlying the Gaussian two-source amplitude which, by contrast, is a trivial problem in its QFT continuum approximation. However, this approach is not intended to replace or augment QFT computations. Rather, our proposed theory X is fundamental to QFT and constitutes a new program for physics, much as quantum physics relates to classical physics. Therefore, the motivation for our theory X is, at this point, conceptual and while there are many conceptual arguments to be made for our approach[7], we restrict ourselves here to the origins of gauge invariance and QFT.

We understand the reader may not be familiar with the path integral formalism, as Healey puts it[8], “While many contemporary physics texts present the path-integral quantization of gauge field theories, and the mathematics of this technique have been intensively studied, I know of no sustained critical discussions of its conceptual foundations.” Therefore, we begin in section 2 with an overview and interpretation of the path integral formalism, which is particularly well-suited for the study of gauge invariance.

2. The Discrete Path Integral Formalism

In this section we provide an overview and interpretation of the path integral approach, showing explicitly how we intend to use “its conceptual foundations.” We employ the discrete path integral formalism because it embodies a 4Dism that allows us to model spacetime matter. For example, the path integral approach is based on the fact that[9] “the [S]ource will emit and the detector receive,” i.e., the path integral formalism deals with Sources and sinks as a unity while invoking a description of the experimental process from initiation to termination. By assuming the discrete path integral is fundamental to the (conventional) continuum path integral, we have a graphical basis for the co-construction of time, space and quantum sources via a self-consistency criterion (SCC). We will show in section 3 how the graphical amalgam of spacetime matter underlies QFT.

2.1. Path Integral in Quantum Physics

In the conventional path integral formalism[10] for non-relativistic quantum mechanics (NRQM) one starts with the amplitude for the propagation from the initial point in configuration space q_I to the final point in configuration space q_F in time T via the unitary operator e^{-iHT} , i.e., $\langle q_F | e^{-iHT} | q_I \rangle$. Breaking the time T into N pieces δt and inserting the identity between each pair of operators $e^{-iH\delta t}$ via the complete set $\int dq |q\rangle \langle q| = 1$ we have

$$\begin{aligned} \langle q_F | e^{-iHT} | q_I \rangle &= \left[\prod_{j=1}^{N-1} \int dq_j \right] \langle q_F | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\delta t} | q_{N-2} \rangle \dots \\ &\quad \langle q_2 | e^{-iH\delta t} | q_1 \rangle \langle q_1 | e^{-iH\delta t} | q_I \rangle. \end{aligned}$$

With $H = \hat{p}^2/2m + V(\hat{q})$ and $\delta t \rightarrow 0$ one can then show that the amplitude is given by

$$\langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t) \exp \left[i \int_0^T dt L(\dot{q}, q) \right], \quad (1)$$

where $L(\dot{q}, q) = m\dot{q}^2/2 - V(q)$. If q is the spatial coordinate on a detector transverse to the line joining Source and detector, then $\prod_{j=1}^{N-1}$ can be thought of as $N-1$ “intermediate” detector surfaces interposed between the Source and the final (real) detector, and $\int dq_j$ can be thought of all possible detection sites on the j^{th} intermediate detector surface. In the continuum limit, these become $\int Dq(t)$ which is therefore viewed as a “sum over all possible paths” from the Source to a particular point on the (real) detector, thus the term “path integral formalism” for conventional NRQM is often understood as a sum over “all paths through space.”

To obtain the path integral approach to QFT one associates q with the oscillator displacement at a *particular point* in space ($V(q) = kq^2/2$). In QFT, one takes the limit $\delta x \rightarrow 0$ so that space is filled with oscillators and the resulting spatial continuity is accounted for mathematically via $q_i(t) \rightarrow q(t, x)$, which is denoted $\phi(t, x)$ and called a “field.” The QFT transition amplitude Z then looks like

$$Z = \int D\phi \exp \left[i \int d^4x L(\dot{\phi}, \phi) \right] \quad (2)$$

where $L(\dot{\phi}, \phi) = (d\phi)^2/2 - V(\phi)$. Impulses J are located in the field to account for particle creation and annihilation; these J are called “sources” in QFT and we have $L(\dot{\phi}, \phi) = (d\phi)^2/2 - V(\phi) + J(t, x)\phi(t, x)$, which can be rewritten as $L(\dot{\phi}, \phi) = \phi D\phi/2 + J(t, x)\phi(t, x)$, where D is a differential operator. In its discrete form (typically, but not necessarily, a hypercubic spacetime lattice), $D \rightarrow \mathbf{K}$ (a difference matrix), $J(t, x) \rightarrow \mathbf{J}$ (each component of which is associated with a point on the spacetime lattice) and $\phi \rightarrow \mathbf{Q}$ (each component of which is associated with a point on the spacetime lattice). Again, part of \mathbf{J} represents field disturbances emanating from a source location (Source) and the other part represents field disturbances incident on a source location (sink) in the conventional view of path integral QFT and, in particle physics, these field disturbances are the particles. We will keep the partition of \mathbf{J} into Sources and sinks in our theory X, but there will be no vacuum lattice structure between the discrete set of sources. The discrete counterpart to (2) is then [11]

$$Z = \int \dots \int dQ_1 \dots dQ_N \exp \left[\frac{i}{2} \mathbf{Q} \cdot \mathbf{K} \cdot \mathbf{Q} + i \mathbf{J} \cdot \mathbf{Q} \right]. \quad (3)$$

In conventional quantum physics, NRQM is understood as $(0+1)$ -dimensional QFT.

2.2. Our Interpretation of the Path Integral in Quantum Physics

We agree that NRQM is to be understood as $(0+1)$ -dimensional QFT, but point out this is at conceptual odds with our derivation of (1) when $\int Dq(t)$ represented a sum over all paths in space, i.e., when q was understood as a location in space (specifically,

a location along a detector surface). If NRQM is $(0 + 1)$ –dimensional QFT, then q is a field displacement at a single location in space. In that case, $\int Dq(t)$ must represent a sum over all field values at a particular point on the detector, not a sum over all paths through space from the Source to a particular point on the detector (sink). So, how *do* we relate a point on the detector (sink) to the Source?

In answering this question, we now explain a formal difference between conventional path integral NRQM and our proposed approach: our links only connect and construct discrete sources \mathbf{J} , there are no source-to-spacetime links (there is no vacuum lattice structure, only spacetime matter). Instead of $\delta x \rightarrow 0$, as in QFT, we assume δx is measureable for (such) NRQM phenomenon. More specifically, we propose starting with (3) whence (roughly) NRQM obtains in the limit $\delta t \rightarrow 0$, as in deriving (1), and QFT obtains in the additional limit $\delta x \rightarrow 0$, as in deriving (2). The QFT limit is well understood as it is the basis for lattice gauge theory and regularization techniques, so one might argue that we are simply *clarifying* the NRQM limit where the path integral formalism is not widely employed. However, again, we are proposing a discrete starting point for theory X, as in (3). Of course, that discrete spacetime is fundamental while “the usual continuum theory is very likely only an approximation[12]” is not new.

2.3. Discrete Path Integral is Fundamental

The version of theory X we propose is a discrete path integral over graphs, so (3) is not a discrete approximation of (1) & (2), but rather (1) & (2) are *continuous approximations of (3)*. In the arena of quantum gravity it is not unusual to find discrete theories[13] that are in some way underneath spacetime theory and theories of “matter” such as QFT, e.g., causal dynamical triangulations[14], quantum graphity[15] and causets[16]. While these approaches are interesting and promising, the approach taken here for theory X will look more like Regge calculus quantum gravity (see Bahr & Dittrich [17] and references therein for recent work along these lines) modified to contain no vacuum lattice structure.

Placing a discrete path integral at bottom introduces conceptual and analytical deviations from the conventional, continuum path integral approach. Conceptually, (1) of NRQM represents a sum over all field values at a particular point on the detector, while (3) of theory X is a mathematical machine that measures the “symmetry” (strength of stationary points) contained in the core of the discrete action

$$\frac{1}{2}\mathbf{K} + \mathbf{J} \tag{4}$$

This core or *actional* yields the discrete action after operating on a particular vector \mathbf{Q} (field). The actional represents a *fundamental/topological, 4D description of the experiment* and Z is a measure of its symmetry. [In its Euclidean form, which is the form we will use, Z is a partition function.] For this reason we prefer to call Z the symmetry amplitude of the 4D experimental configuration. Analytically, because we are *starting* with a discrete formalism, we are in position to mathematically explicate

trans-temporal identity, whereas this process is unarticulated elsewhere in physics. As we will now see, this leads to our proposed self-consistency criterion (SCC) underlying Z .

2.4. Self-Consistency Criterion

Our use of a self-consistency criterion is not without precedent, as we already have an ideal example in Einstein's equations of general relativity (GR). Momentum, force and energy all depend on spatiotemporal measurements (tacit or explicit), so the stress-energy tensor cannot be constructed without tacit or explicit knowledge of the spacetime metric (technically, the stress-energy tensor can be written as the functional derivative of the matter-energy Lagrangian with respect to the metric). But, if one wants a "dynamic spacetime" in the parlance of GR, the spacetime metric must depend on the matter-energy distribution in spacetime. GR solves this dilemma by demanding the stress-energy tensor be "consistent" with the spacetime metric per Einstein's equations. For example, concerning the stress-energy tensor, Hamber and Williams write[18], "In general its covariant divergence is not zero, but consistency of the Einstein field equations demands $\nabla^\alpha T_{\alpha\beta} = 0$." This self-consistency hinges on divergence-free sources, which finds a mathematical underpinning in $\partial\partial = 0$. So, Einstein's equations of GR are a mathematical articulation of the boundary of a boundary principle at the classical level, i.e., they constitute a self-consistency criterion at the classical level, as are quantum and classical electromagnetism[19]. We will provide an explanation for this fact in section 3, but essentially the graphical SCC of our theory X gives rise to continuum counterparts in QFT and classical field theory.

In order to illustrate the discrete mathematical co-constuction of space, time and sources \mathbf{J} , we will use graph theory a la Wise[20] and find that $\partial_1 \cdot \partial_1^T$, where ∂_1 is a boundary operator in the spacetime chain complex of our graph satisfying $\partial_1 \cdot \partial_2 = 0$, has precisely the same form as the difference matrix in the discrete action for coupled harmonic oscillators. Therefore, we are led to speculate that $\mathbf{K} \propto \partial_1 \cdot \partial_1^T$. Defining the source vector \mathbf{J} relationally via $\mathbf{J} \propto \partial_1 \cdot \mathbf{e}$ then gives tautologically per $\partial_1 \cdot \partial_2 = 0$ both a divergence-free \mathbf{J} and $\mathbf{K} \cdot \mathbf{v} \propto \mathbf{J}$, where \mathbf{e} is the vector of links and \mathbf{v} is the vector of vertices. $\mathbf{K} \cdot \mathbf{v} \propto \mathbf{J}$ is our SCC following from $\partial_1 \cdot \partial_2 = 0$, and it defines what is meant by a self-consistent co-construction of space, time and divergence-free sources \mathbf{J} , thereby constraining \mathbf{K} and \mathbf{J} in Z . Thus, our SCC provides a basis for the discrete action and supports our view that (3) is fundamental to (1) & (2), rather than the converse. Conceptually, that is the basis of our discrete, graphical path integral approach to theory X. We now provide the details.

3. The Formalism

3.1. The General Approach

Again, in theory X, the symmetry amplitude Z contains a discrete action constructed per a self-consistency criterion (SCC) for space, time and divergence-free sources \mathbf{J} . As introduced in section 2 and argued later in this section, we will codify the SCC using \mathbf{K} and \mathbf{J} ; these elements are germane to the transition amplitude Z in the Central Identity of Quantum Field Theory[21],

$$Z = \int D\phi \exp \left[-\frac{1}{2} \phi \cdot \mathbf{K} \cdot \phi - V(\phi) + \mathbf{J} \cdot \phi \right] = \exp \left[-V \left(\frac{\delta}{\delta J} \right) \right] \exp \left[\frac{1}{2} \mathbf{J} \cdot \mathbf{K}^{-1} \cdot \mathbf{J} \right]. \quad (5)$$

While the field is a mere integration variable used to produce Z , it must reappear at the level of classical field theory. To see how the field makes its appearance per theory X, consider (5) for the simple Gaussian theory ($V(\phi) = 0$). On a graph with N vertices, (5) is

$$Z = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dQ_1 \dots dQ_N \exp \left[-\frac{1}{2} \mathbf{Q} \cdot \mathbf{K} \cdot \mathbf{Q} + \mathbf{J} \cdot \mathbf{Q} \right] \quad (6)$$

with a solution of

$$Z = \left(\frac{(2\pi)^N}{\det \mathbf{K}} \right)^{1/2} \exp \left[\frac{1}{2} \mathbf{J} \cdot \mathbf{K}^{-1} \cdot \mathbf{J} \right]. \quad (7)$$

It is easiest to work in an eigenbasis of \mathbf{K} and (as will argue later) we restrict the path integral to the row space of \mathbf{K} , this gives

$$Z = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} d\tilde{Q}_1 \dots d\tilde{Q}_{N-1} \exp \left[\sum_{j=1}^{N-1} \left(-\frac{1}{2} \tilde{Q}_j^2 a_j + \tilde{J}_j \tilde{Q}_j \right) \right] \quad (8)$$

where \tilde{Q}_j are the coordinates associated with the eigenbasis of \mathbf{K} and \tilde{Q}_N is associated with eigenvalue zero, a_j is the eigenvalue of \mathbf{K} corresponding to \tilde{Q}_j , and \tilde{J}_j are the components of \mathbf{J} in the eigenbasis of \mathbf{K} . The solution of (8) is

$$Z = \left(\frac{(2\pi)^{N-1}}{\prod_{j=1}^{N-1} a_j} \right)^{1/2} \prod_{j=1}^{N-1} \exp \left(\frac{\tilde{J}_j^2}{2a_j} \right). \quad (9)$$

On our view, the experiment is described fundamentally by \mathbf{K} and \mathbf{J} on our topological graph. Again, per (9), there is no field \tilde{Q} appearing in Z at this level, i.e., \tilde{Q} is only an integration variable. \tilde{Q} makes its first appearance as something more than an integration variable when we produce probabilities from Z . That is, since we are working with a Euclidean path integral, Z is a partition function and the probability of measuring $\tilde{Q}_k = \tilde{Q}_0$ is found by computing the fraction of Z which contains \tilde{Q}_0 at the k^{th} vertex[22]. We have

$$P(\tilde{Q}_k = \tilde{Q}_0) = \frac{Z(\tilde{Q}_k = \tilde{Q}_0)}{Z} = \sqrt{\frac{a_k}{2\pi}} \exp \left(-\frac{1}{2} \tilde{Q}_0^2 a_k + \tilde{J}_k \tilde{Q}_0 - \frac{\tilde{J}_k^2}{2a_k} \right) \quad (10)$$

as the part of theory X approximated in the continuum by QFT. The most probable value of \tilde{Q}_0 at the k^{th} vertex is then given by

$$\delta P(\tilde{Q}_k = \tilde{Q}_0) = 0 \implies \delta \left(-\frac{1}{2} \tilde{Q}_0^2 a_k + \tilde{J}_k \tilde{Q}_0 - \frac{\tilde{J}_k^2}{2a_k} \right) = 0 \implies a_k \tilde{Q}_0 = \tilde{J}_k. \quad (11)$$

That is, $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$ is the part of theory X that obtains statistically and is approximated in the continuum by classical field theory. We note that the manner by which $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$ follows from $P(\tilde{Q}_k = \tilde{Q}_0) = Z(\tilde{Q}_k = \tilde{Q}_0)/Z$ parallels the manner by which classical field theory follows from QFT via the stationary phase method[23]. Thus, one may obtain classical field theory by the continuum limit of $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$ in theory X (theory X \rightarrow classical field theory), or by first obtaining QFT via the continuum limit of $P(\tilde{Q}_k = \tilde{Q}_0) = Z(\tilde{Q}_k = \tilde{Q}_0)/Z$ in theory X and then by using the stationary phase method on QFT (theory X \rightarrow QFT \rightarrow classical field theory). In either case, QFT is not quantized classical field theory in our approach. In summary:

- (i) Z is a partition function for an experiment described topologically by $\mathbf{K}/2 + \mathbf{J}$ (Figure 1a).
- (ii) $P(\tilde{Q}_k = \tilde{Q}_0) = Z(\tilde{Q}_k = \tilde{Q}_0)/Z$ gives us the probability for a particular geometric outcome in that experiment (Figures 1b and 2b).
- (iii) $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$ gives us the most probable values of the experimental outcomes which are then averaged to produce the geometry for the experimental procedure at the classical level (Figure 2a).
- (iv) $P(\tilde{Q}_k = \tilde{Q}_0) = Z(\tilde{Q}_k = \tilde{Q}_0)/Z$ and $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$ are the parts of theory X approximated in the continuum by QFT and classical field theory, respectively.

3.2. The Two-Source Euclidean Symmetry Amplitude/Partition Function

Typically, one identifies fundamentally interesting physics with symmetries of the action in the Central Identity of Quantum Field Theory, but we have theory X fundamental to QFT, so our method of choosing fundamentally interesting physics must reside in the topological graph of theory X. Thus, we seek a constraint of \mathbf{K} and \mathbf{J} in our graphical symmetry amplitude Z and this will be in the form of a self-consistency criterion (SCC). In order to motivate our general method, we will first consider a simple graph with six vertices, seven links and two plaquettes for our $(1+1)$ -dimensional spacetime model (Figure 3). Our goal with this simple model is to seek relevant structure that might be used to infer an SCC. We begin by constructing the boundary operators over our graph.

The boundary of \mathbf{p}_1 is $\mathbf{e}_4 + \mathbf{e}_5 - \mathbf{e}_2 - \mathbf{e}_1$, which also provides an orientation. The boundary of \mathbf{e}_1 is $\mathbf{v}_2 - \mathbf{v}_1$, which likewise provides an orientation. Using these conventions for the orientations of links and plaquettes we have the following boundary operator for $C_2 \rightarrow C_1$, i.e., space of plaquettes mapped to space of links in the spacetime chain

complex:

$$\partial_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad (12)$$

Notice the first column is simply the links for the boundary of \mathbf{p}_1 and the second column is simply the links for the boundary of \mathbf{p}_2 . We have the following boundary operator for $C_1 \rightarrow C_0$, i.e., space of links mapped to space of vertices in the spacetime chain complex:

$$\partial_1 = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad (13)$$

which completes the spacetime chain complex, $C_0 \leftarrow C_1 \leftarrow C_2$. Notice the columns are simply the vertices for the boundaries of the edges. These boundary operators satisfy $\partial_1 \cdot \partial_2 = 0$, i.e., the boundary of a boundary principle.

The potential for coupled oscillators can be written

$$V(q_1, q_2) = \sum_{a,b} \frac{1}{2} k_{ab} q_a q_b = \frac{1}{2} k q_1^2 + \frac{1}{2} k q_2^2 + k_{12} q_1 q_2 \quad (14)$$

where $k_{11} = k_{22} = k > 0$ and $k_{12} = k_{21} < 0$ per the classical analogue (Figure 4) with $k = k_1 + k_3 = k_2 + k_3$ and $k_{12} = -k_3$ to recover the form in (14). The Lagrangian is then

$$L = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 - \frac{1}{2} k q_1^2 - \frac{1}{2} k q_2^2 - k_{12} q_1 q_2 \quad (15)$$

so our NRQM Euclidean symmetry amplitude is

$$Z = \int Dq(t) \exp \left[- \int_0^T dt \left(\frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 + V(q_1, q_2) - J_1 q_1 - J_2 q_2 \right) \right] \quad (16)$$

after Wick rotation. This gives

$$\mathbf{K} = \begin{bmatrix} \left(\frac{m}{\Delta t} + k \Delta t \right) & -\frac{m}{\Delta t} & 0 & k_{12} \Delta t & 0 & 0 \\ -\frac{m}{\Delta t} & \left(\frac{2m}{\Delta t} + k \Delta t \right) & -\frac{m}{\Delta t} & 0 & k_{12} \Delta t & 0 \\ 0 & -\frac{m}{\Delta t} & \left(\frac{m}{\Delta t} + k \Delta t \right) & 0 & 0 & k_{12} \Delta t \\ k_{12} \Delta t & 0 & 0 & \left(\frac{m}{\Delta t} + k \Delta t \right) & -\frac{m}{\Delta t} & 0 \\ 0 & k_{12} \Delta t & 0 & -\frac{m}{\Delta t} & \left(\frac{2m}{\Delta t} + k \Delta t \right) & -\frac{m}{\Delta t} \\ 0 & 0 & k_{12} \Delta t & 0 & -\frac{m}{\Delta t} & \left(\frac{m}{\Delta t} + k \Delta t \right) \end{bmatrix} \quad (17)$$

on our graph. Thus, we borrow (loosely) from Wise[24] and suggest $\mathbf{K} \propto \partial_1 \cdot \partial_1^T$ since

$$\partial_1 \cdot \partial_1^T = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix} \quad (18)$$

produces precisely the same form as (17) and quantum theory is known to be “rooted in this harmonic paradigm[25].” [In fact, these matrices will continue to have the same form as one increases the number of vertices in Figure 3.] Now we construct a suitable candidate for \mathbf{J} , relate it to \mathbf{K} and infer our SCC.

Recall that \mathbf{J} has a component associated with each vertex so here it has components, J_n , $n = 1, 2, \dots, 6$; J_n for $n = 1, 2, 3$ represents one source and J_n for $n = 4, 5, 6$ represents the second source. We propose $\mathbf{J} \propto \partial_1 \cdot \mathbf{e}$, where e_i are the links of our graph, since

$$\partial_1 \cdot \mathbf{e} = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = \begin{bmatrix} -e_1 - e_4 \\ e_1 - e_2 - e_3 \\ e_3 - e_7 \\ e_4 - e_5 \\ e_2 + e_5 - e_6 \\ e_6 + e_7 \end{bmatrix} \quad (19)$$

automatically makes \mathbf{J} divergence-free, i.e., $\sum_i J_i = 0$. Such a relationship on discrete spacetime lattices is not new. For example, Sorkin showed that charge conservation follows from gauge invariance for the electromagnetic field on a simplicial net[26].

With these definitions of \mathbf{K} and \mathbf{J} we have, ipso facto, $\mathbf{K} \cdot \mathbf{v} \propto \mathbf{J}$ as the basis of our SCC since

$$\partial_1 \cdot \partial_1^T \cdot \mathbf{v} = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} -e_1 - e_4 \\ e_1 - e_2 - e_3 \\ e_3 - e_7 \\ e_4 - e_5 \\ e_2 + e_5 - e_6 \\ e_6 + e_7 \end{bmatrix} = \partial_1 \cdot \mathbf{e} \quad (20)$$

where we have used $e_1 = v_2 - v_1$ (etc.) to obtain the last column. You can see that the boundary of a boundary principle underwrites (20) by the definition of “boundary” and from the fact that the links are directed and connect one vertex to another, i.e., they do not start or end ‘off the graph’. Likewise, this fact and our definition of \mathbf{J} imply $\sum_i J_i = 0$, which is our graphical equivalent of a divergence-free, relationally defined source (every link leaving one vertex goes into another vertex). Thus, the SCC

$\mathbf{K} \cdot \mathbf{v} \propto \mathbf{J}$ and divergence-free sources $\sum_i J_i = 0$ obtain tautologically via the boundary of a boundary principle. The SCC also guarantees that \mathbf{J} resides in the row space of \mathbf{K} so, as will be shown, we can avoid having to “throw away infinities” associated with gauge groups of infinite volume as in Faddeev-Popov gauge fixing. \mathbf{K} has at least one eigenvector with zero eigenvalue which is responsible for gauge invariance, so *the self-consistent co-construction of space, time and divergence-free sources entails gauge invariance*.

Moving now to N dimensions, the Wick rotated version of (3) is (6) and the solution is (7). Using $\mathbf{J} = \alpha \partial_1 \cdot \mathbf{e}$ and $\mathbf{K} = \beta \partial_1 \cdot \partial_1^T$ ($\alpha, \beta \in \mathbb{R}$) with the SCC gives $\mathbf{K} \cdot \mathbf{v} = (\beta/\alpha) \mathbf{J}$, so that $\mathbf{v} = (\beta/\alpha) \mathbf{K}^{-1} \cdot \mathbf{J}$. However, \mathbf{K}^{-1} does not exist because \mathbf{K} has a nontrivial null space, therefore the row space of \mathbf{K} is an $(N - 1)$ -dimensional subspace of the N -dimensional vector space[§]. The eigenvector with eigenvalue of zero, i.e., normal to this hyperplane, is $\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \end{bmatrix}^T$, which follows from the SCC as shown supra. Since \mathbf{J} resides in the row space of \mathbf{K} and, on our view, Z is a functional of \mathbf{K} and \mathbf{J} which produces a partition function for the various $\mathbf{K}/2 + \mathbf{J}$ associated with different 4D experimental configurations, we restrict the path integral of (6) to the row space of \mathbf{K} . Thus, our approach revises (7) to give (9).

We find in general that half the eigenvectors of \mathbf{K} are of the form $\begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix}$ and half are of the form $\begin{bmatrix} \mathbf{x} \\ -\mathbf{x} \end{bmatrix}$. The eigenvalues are given by $\lambda \pm 1$ where $\lambda - 1$ is the eigenvalue for $\begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix}$, $\lambda + 1$ is the eigenvalue for $\begin{bmatrix} \mathbf{x} \\ -\mathbf{x} \end{bmatrix}$, and $\lambda_j = 3 - 2 \cos(2j\pi/N)$, $j = 0, \dots, N/2 - 1$. The k components of \mathbf{x} for a given λ_j are $x_{jk} = \sqrt{\frac{2}{N}} \cos\left(\frac{j(2k-1)\pi}{N}\right)$, $k = 1, \dots, N/2$ for $j > 0$ and $x_{0k} = 1/\sqrt{N}$, $k = 1, \dots, N/2$ for $j = 0$ ($j = 0 \rightarrow$ eigenvalues of \mathbf{K} are 0 and 2). As you can see, there are no degeneracies within the $\begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix}$ subspace or the $\begin{bmatrix} \mathbf{x} \\ -\mathbf{x} \end{bmatrix}$ subspace. Therefore, the only degeneracies occur between subspaces, so we know all degenerate eigenvalues are associated with unique eigenvectors, as alluded to in a previous footnote.

We have N vertices and $(3N/2 - 2)$ links. Define the temporal (vertical) links e_i in terms of vertices v_i in the following fashion: $e_i = v_{i+1} - v_i$, $i = 1, \dots, N/2 - 1$ and $e_{N/2+i-1} = v_{N/2+i+1} - v_{N/2+i}$, $i = 1, \dots, N/2 - 1$. Define the spatial (horizontal) links

[§] This assumes the number of degenerate eigenvalues always equals the dimensionality of the subspace spanned by their eigenvectors, which we will see is true for \mathbf{K} in this example.

via: $e_{N+i-2} = v_{N/2+i} - v_i$, $i = 1, \dots, N/2$. This gives

$$\mathbf{J} = \begin{bmatrix} -e_1 - e_{N-1} \\ -e_i + e_{i-1} - e_{N+i-2} & i = 2, \dots, \frac{N}{2} - 1 \\ e_{N/2-1} - e_{N+N/2-2} \\ e_{N-1} - e_{N/2} \\ e_{N/2+i-2} + e_{N+i-2} - e_{N/2+i-1} & i = 2, \dots, \frac{N}{2} - 1 \\ e_{N+N/2-2} + e_{N-2} \end{bmatrix}. \quad (21)$$

We then need to find the projection of \mathbf{J} on each of the orthonormal eigenvectors of \mathbf{K} that have non-zero eigenvalues. Call each projection $\tilde{J}_i = \langle i | J \rangle$, where $\langle i |$ is the i^{th} orthonormal eigenvector. Let a_i ($i = 1, \dots, N-1$) be the non-zero eigenvalues of \mathbf{K} associated with the eigenvectors $\langle i |$, ($i = 1, \dots, N-1$), respectively. To complete the two-source Euclidean symmetry amplitude we need to compute the exponent

$$\Phi = \sum_{i=1}^{N-1} \frac{(\tilde{J}_i)^2}{2a_i \hbar \beta} \quad (22)$$

where \hbar is viewed as a fundamental scaling factor with the dimensions of action. We find $\Phi = (\Phi_S + \Phi_T + \Phi_{ST})/(2\hbar\beta)$, where

$$\Phi_S = \frac{2\alpha^2}{N} \left(\sum_{k=1}^{N/2} e_{k+N-2} \right)^2 \quad (23)$$

involves only spatial links

$$\Phi_T = \frac{2\alpha^2}{N} \sum_{j=1}^{N/2-1} \left[\sum_{k=1}^{N/2-1} (e_k + e_{k+N/2-1}) \sin \left(\frac{2jk\pi}{N} \right) \right]^2 \quad (24)$$

involves only temporal links and

$$\begin{aligned} \Phi_{ST} = \sum_{j=1}^{N/2-1} \frac{4\alpha^2}{N(1 + 2\sin^2(\frac{j\pi}{N}))} & \left[\sin \left(\frac{j\pi}{N} \right) \sum_{k=1}^{N/2-1} (e_k - e_{k+N/2-1}) \sin \left(\frac{2jk\pi}{N} \right) \right. \\ & \left. + \sum_{k=1}^{N/2} e_{k+N-2} \cos \left(\frac{(2k-1)j\pi}{N} \right) \right]^2 \end{aligned} \quad (25)$$

involves a mix of spatial and temporal links. (23) comes from the eigenvalue 2 associated with $\begin{bmatrix} \mathbf{x} \\ -\mathbf{x} \end{bmatrix}$, which exists for all N under consideration. (25) comes from the remaining

eigenvalues associated with $\begin{bmatrix} \mathbf{x} \\ -\mathbf{x} \end{bmatrix}$. (24) comes from the eigenvalues associated with $\begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix}$ having omitted zero, which exists for all N under consideration.

4. Conclusion

We have assumed the existence of a discrete theory (X) fundamental to quantum physics, the characteristics of which we articulated and explored via a path integral formalism over graphs. Mathematically, one can summarize our proposed theory X as follows:

$$\mathbf{K} \cdot \mathbf{v} \propto \mathbf{J} \rightarrow \frac{1}{2}\mathbf{K} + \mathbf{J} \rightarrow Z \rightarrow P\left(\tilde{Q}_k = \tilde{Q}_0\right) = \frac{Z\left(\tilde{Q}_k = \tilde{Q}_0\right)}{Z} \rightarrow \mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$$

with QFT and classical field theory understood as the continuum approximations to $P\left(\tilde{Q}_k = \tilde{Q}_0\right) = \frac{Z\left(\tilde{Q}_k = \tilde{Q}_0\right)}{Z}$ and $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$, respectively. Thus, the graphical SCC $\mathbf{K} \cdot \mathbf{v} \propto \mathbf{J}$ statistically reproduces its counterpart $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$ whence classical field theory. While the mathematical details of theory X provided herein are too simplistic to unify physics formally, we do believe they provide a respectable conceptual response to Glashow and Rovelli's challenges presented in section 1. Our proposed new approach to theory X underlying QFT accounts naturally for gauge invariance via a self-consistency criterion and deals effectively with factors of infinity associated with gauge groups of infinite volume, since the transition amplitude Z is restricted to the row space of the difference matrix and source vector.

While positing a discrete theory at bottom is hardly unique in fundamental physics, and our formal development is tentative, our overall approach to theory X is novel in that it is adynamical and acausal, in contrast to other fundamental theories such as M-theory, loop quantum gravity, causets, etc. Such theories may deviate from the norm by employing radical new fundamental entities (branes, loops, ordered sets, etc.), but the game is always dynamical, broadly construed (vibrating branes, geometrodynamics, sequential growth process, etc.). While itself adynamical, the SCC guarantees the graph will produce divergence-free classical dynamics in the appropriate statistical and continuum limits, and provides an acausal global constraint that results in a self-consistent, co-construction of space, time and matter that is *de facto* background independent. Thus in our approach, one has an acausal, adynamical unity of “spacetime+matter” at the fundamental level that results statistically in the causal, dynamical “spacetime + matter” of classical physics. Consequently, fundamental explanation is in terms of a global, adynamical organizing principle. And, ultimate explanation in physics is not in terms of some thing or dynamical entity (obeying a new dynamical equation) “at the bottom” conceived at higher energies and smaller spatiotemporal scales, begging for justification from something at some yet “deeper” scale, but self-consistency writ large for the explanatory “process” as a whole.

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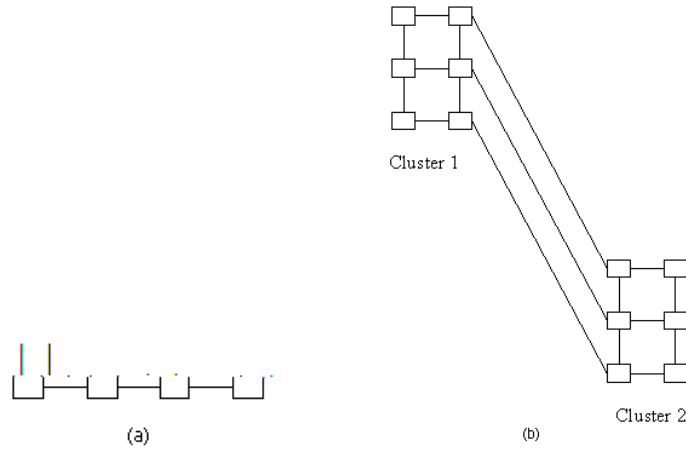


Figure 1. (a) Topological Graph - This spacetime matter graph depicts four sources, i.e., the columns of squares. The graph's action $\mathbf{K}/2 + \mathbf{J}$, such that $\mathbf{K} \cdot \mathbf{v} \propto \mathbf{J}$, characterizes the graphical topology, which underwrites a partition function Z for spatiotemporal geometries over the graph. (b) Geometric Graph - The topological graph of (a) is endowed with a particular distribution of spatiotemporal geometric relations, i.e., link lengths as determined by the field values Q on their respective vertices. Clusters 1 & 2 are the result of this geometric process for a particular distribution of field values Q .

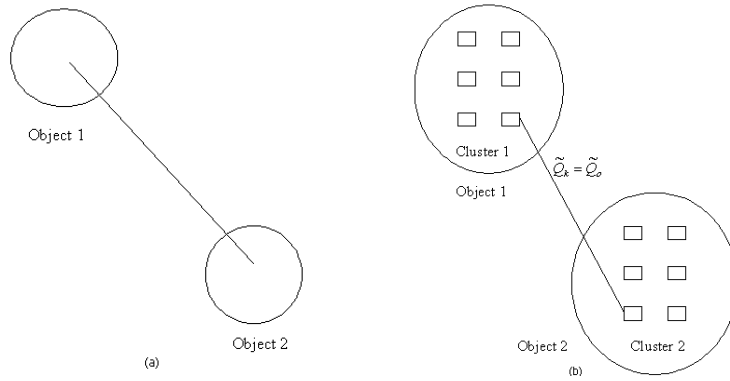


Figure 2. (a) Classical Physics - Classical Objects result when the most probable field values \mathbf{Q}_0 yield spatiotemporally localized Clusters 1 & 2 as in Figure 1b. The lone link in this figure represents the average of the link lengths obtained via the most probable field values \mathbf{Q}_0 . The most probable values \mathbf{Q}_0 are found via $\mathbf{K} \cdot \mathbf{Q}_0 = \mathbf{J}$, so this is the origin of classical physics. (b) Quantum Physics - A particular outcome \tilde{Q}_0 of a quantum physics experiment allows one to compute the k^{th} link length of the geometric graph in the context of the classical Objects comprising the experiment, e.g., Source, beam splitters, mirrors, and detectors. The partition function provides the probability of this particular outcome, i.e., $P(\tilde{Q}_k = \tilde{Q}_0) = \frac{Z(\tilde{Q}_k = \tilde{Q}_0)}{Z}$.

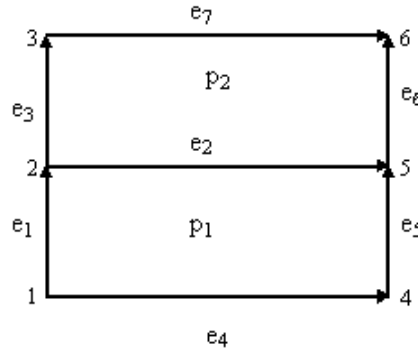


Figure 3. Graph with six vertices, seven links e_i and two plaquettes p_i .

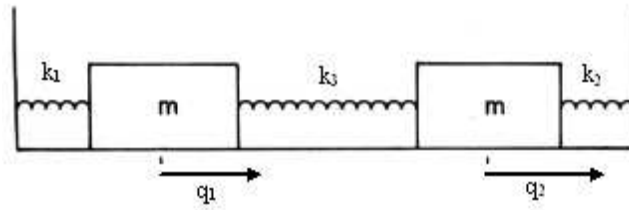


Figure 4. Coupled harmonic oscillators.

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